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The Bicategory of m-regular Involutive Quantales

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It is well known that rings are the objects of a bicategory, whose arrows are bimodules, composed through the bimodule tensor product. We give an analogous bicategorical description of m-regular involutive quantales. The upshot is that known definition of Morita equivalence for this case amounts to isomorphism of objects in the pertinent bicategory.

KEY WORDS: bicategory; bimodules; bimodule tensor product; m-regular involutive quantale; Morita equivalence.

1. INTRODUCTION

Recently the theory of Morita equivalence for involutive quantales and the notion of the interior tensor products of Hilbert modules over involutive quantales evolved considerably (see e.g. Paseka, 2002 and Paseka, 2001). The present paper is an attempt to put a part of this theory in a broader context of the bicategory of m-regular involutive quantales. For facts concerning quantales in general we refer to Rosenthal (1990), for definitions and motivation concerning involutive quantales we recommend Mulvey and Pelletier (1992).

A version of Morita's theory appropriate for C*-algebras was developed by Rieffel (see Rieffel, 1974 and Raeburn and Williams, 1998) and Blecher (see Blecher, 2001). It is also well known, for example (see Landsman, 2001a and Landsman, 2001b), that C*-algebras form a bicategory and two C*-algebras *A, B* are isomorphic in this bicategory iff they are Morita equivalent. Here we shall present a translation of this result to m-regular involutive quantales.

The plan is as follows: in Section 2 we shall review some basic facts concerning Hilbert modules and modules over involutive quantales in general and then we recall the notion of a bicategory. In Section 3 we subsequently explain the bicategory of m-regular involutive quantales in some detail, including the pertinent Morita theory.

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2. PRELIMINARIES

Let us begin by establishing the common symbols and notations in this paper. Let A be an involutive quantale. We shall denote by MOD_A ($_AMOD$) the category of right (left) A–modules with the right (left) action $\diamond$ ($\bullet$) and right (left) module homomorphisms. For a right A-module X in MOD_A the submodule $\mathrm{ess}(X) = X \diamond A$ generated by the elements $x \diamond a$ is called the *essential part* of X. If $\mathrm{ess}(X) = X$ we say that X is (right) *essential*. We shall say that A is (*right*) *separating* for the A-module M and that M is (*right*) *separated* by A if $m\diamond (—) = n\diamond (—)$ implies $m = n$. We say that M is *m-regular* if it is both separated by A and essential. An involutive quantale A is called *m-regular* if it is m-regular as an A-module. Then, evidently $1 \cdot 1 = 1$ in A and $a\cdot (—) = b\cdot (—)$ implies $a = b$. In what follows we shall always assume that our involutive quantales will be m-regular.

Note that we may dually define the notion of a left (essential, separating, m-regular) A-module with a left multiplication $\bullet$. Evidently, A is m-regular iff it is left m-regular.

The theory of right (left) Hilbert A-modules (we refer the reader to Paseka, 1999 for details and examples) is a generalization of the theory of complete semilattices with a duality and it is the natural framework for the study of modules over an involutive quantale A endowed with A-valued inner products. Note only that for each right Hilbert A-module M there is a *conjugate* left Hilbert A-module M^*: as an sup-semilattice M^* is just M with a left module multiplication $a \bullet m = m \diamond a^*$ and a left A-valued inner product $_A\langle m, n\rangle = \langle m, n\rangle_A$ (see Lemma 1.5 in Paseka, 1999). Recall that a Hilbert A-module M is said to be *full* if $\langle M = M\rangle = \{\bigvee_{j\in J}\langle x_j, y_j\rangle : x_j, y_j \in M, j \in J\} = A$. Evidently, any conjugate Hilbert module to an m-regular (full) Hilbert module is m-regular (full).

Let $f : M \to N$ be a map between right (left) Hilbert A-modules. We say that a map $g : N \to M$ is *a^*-adjoint to f* and f is *adjointable* if $\langle f(m), n\rangle = \langle m, g(n)\rangle$ for all $m \in M, n \in N$. Note that the *-adjoint g to f is uniquely determined. Namely, if $\langle f(m), n\rangle = \langle m, g(n)\rangle = \langle m, h(n)\rangle$ then $g(n) = h(n)$ for all n i.e. $g = h$. We then put $f^* = g$. Evidently, any adjointable map is a right (left) module homomorphism (see Paseka, 1999). The set of all adjointable maps from M to N is denoted by $\mathcal{A}_A(M, N)$, $\mathcal{A}_A(M, M) = \mathcal{A}_A(M, M)$. The *representation category* $\mathrm{Rep}(A)$ of an m-regular involutive quantale A has m-regular right Hilbert A-modules as objects, and adjointable maps as arrows.

Let B be an m-regular involutive quantale such that $\varphi : B \to \mathcal{A}_A(M)$ is an involutive quantale homomorphism. Then φ is said to be *nondegenerate* if $M = \varphi(B(M)$. i.e. for any $m \in M$ we have that $m = \bigvee_{j\in J}\varphi(b_j)(m_j)$ for suitable elements $m_j \in M$ and $b_j \in B$, $j \in J$. Similarly, an involutive quantale homomorphism $f : B \to A$ is said to be *nondegenerate* if the left B-module A with the action $b \bullet a = f(b) \cdot a$ is essential.

We say that an adjointable map $f : M \to N$ is an *isometry* if, for all $m_1, m_2 \in M$, $\langle m_1, m_2 \rangle = \langle f(m_1), f(m_2) \rangle$. This is equivalent to $f^* \circ f = \mathrm{id}_M$. Similarly, an adjointable map $f : M \to N$ is *unitary* if $f^* \circ f = \mathrm{id}_M$ and $f \circ f^* = \mathrm{id}_N$. Note that an isometry is unitary iff it is surjective.

Recall that from Paseka (1999) we know that, for any right Hilbert A-module M and for all $m \in M$, the map $m^{\sim} : A \to M$ defined by $a \to m \diamond$ a has a *-adjoint $m^{\star} : M \to A$ defined by $n \mapsto \langle m, n \rangle$. The maps of the form

$$T = \bigvee_{j \in J} \Theta_{n_j}, m_j,$$

here $\Theta_{n,m} = n \diamond \langle m, - \rangle = n^{\sim} \circ m^{\star}$, $m \in M$, $n \in N$ are said to be *compact maps* and the set of all compact maps will be denoted $\mathcal{K}_A(M, N)$ or just $\mathcal{K}_A(M)$ in case $M = N$. Note that any compact map is an adjointable map and that a composition of a compact map with an adjointable one is again compact, i.e. $\mathcal{K}_A(N, P) \circ \mathcal{A}_A(M, N) \subseteq \mathcal{K}_A(M, P)$ and $\mathcal{A}_A(N, P) \circ \mathcal{K}_A(M, N) \subseteq \mathcal{K}_A(M, P)$. An expository treatment of compact maps on Hilbert A-modules may be found in Paseka (1999).

Bicategories generalize categories, as follows.

Definition 2.1. A bicategory $\mathcal{C}$ consists of

- A class of objects $\mathcal{C}_0$ (with elements 0-cells $A, B, \ldots$);
- A category (A, B) (with objects 1-cells $f, g, \ldots$ and arrows 2-cells $\alpha, \beta, \ldots$) for each pair $(A, B) \in \mathcal{C}_0 \times \mathcal{C}_0$;
- A (bi)functor $\phi_{A,B,C} : (A, B) \times (B, C) \to (A, C)$ for each triple $(A, B, C) \in \mathcal{C}_0 \times \mathcal{C}_0 \times \mathcal{C}_0$;
- A functor $U_A : \mathbf{1} \to (A, A)$ from the trivial (or terminal category) $\mathbf{1}$ (with one object and one arrow), $U_A(1) = 1_A$, for each $A \in \mathcal{C}_0$,

such that

1. The functors $\phi_{A,C,D}\phi_{A,B,C}$ and $\phi_{A,B,D}\phi_{B,C,D}$ from $(A, B) \times (B, C) \times (C, D)$ to (A, D) are naturally isomorphic;
2. The functors $f \mapsto f 1_B$ (where $f \in (A, B)$) and $\mathrm{id}_{(A,B)}$ from (A, B) to itself are naturally isomorphic;
3. The functors $1_A f \mapsto f$ and $\mathrm{id}_{(A,B)}$ from (A, B) to itself are naturally isomorphic,

subject to coherence laws stated on p. 282 of MacLane (1998) (these laws lead to consistency of various orders of bracketing).

The bifunctors ϕ are sometimes said to define the "horizontal" composition of arrows, in contradistinction to the "vertical" composition of arrows in each of the categories (A, B).

Since the theory of Morita equivalence will involve isomorphism of objects in a bicategory, we have to point out that this notion is weaker than in a category.

Definition 2.2. Two objects A, B in a bicategory are isomorphic, written $A \cong B$, when there exist arrows $f \in (A, B)$ and $g \in (B, A)$ such that $fg \simeq 1_A$ (isomorphism in the usual sense as objects in the category (A, A)) and $gf \simeq 1_B$ in (B, B).

3. THE BICATEGORY OF M-REGULAR INVOLUTIVE QUANTALES

The involutive quantale analogue of a bimodule for rings is a Hilbert bimodule. This concept involves the theory of Hilbert modules, for which we refer to Paseka (1999).

Definition 3.1. Let A and B be involutive quantales, and let F and X be right Hilbert B-modules. We say that F is a right *Hilbert $A - B$ bimodule* if it is a left A-module satisfying

$$a \bullet (x \diamond b) = (a \bullet x) \diamond b \text{ and } \langle a \bullet x, y\rangle_B = \langle x, a^* \bullet y\rangle_B$$

for all $a \in A, x, y \in F$, and $b \in B$. We say that F is an m-regular right Hilbert $A - B$ bimodule if it is both an essential left A-module and an m-regular right B-module. We write then $A \rightarrow F \rightleftharpoons B$.

We say that a (full m-regular) right Hilbert $A - B$ bimodule X is a Hilbert (imprimitivity) $A - B$ bimodule if X is also a (full m-regular) left Hilbert A-module in such a way that

$$_A\langle x, y\rangle \bullet z = x \diamond \langle y, z\rangle_B \text{ and } _A\langle x \diamond b, y\rangle = {}_A\langle x, y \diamond b^*\rangle.$$

The following example is the involutive quantale version of the ring bimodule $R \rightarrow R \rightleftharpoons R$.

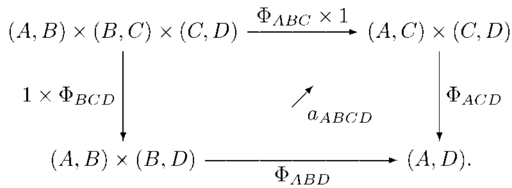
Example 3.2. An m-regular involutive quantale A may be seen as an imprimitivity Hilbert $A - A$ bimodule $A \rightarrow A \rightleftharpoons A$ over itself, in which $\langle a, b\rangle_A = a^*b$, and the left and right actions are given by left and right multiplication, respectively.

The involutive quantale analogue of the ring bimodule tensor product is the interior tensor product $\dot{\otimes}_B$ defined and studied in Paseka (2001). Recall only that a map that is both a left module map and a right adjointable map is preserved by an interior tensor product. In complete parallel with ring theory and C^*-algebra theory, one now has

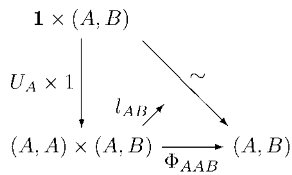
Theorem 3.3. *For any two m-regular involutive quantales A, B, let (A, B) be the collection of all m-regular right Hilbert $A - B$ bimodules $A \rightarrow M \rightleftharpoons B$, seen as a category, whose arrows are adjointable A-module maps (such maps are automatically bimodule maps).*

With (horizontal) composition functor $\phi_{ABC} : (A, B) \times (B, C) \to (A, C)$ given by $\dot{\otimes}_B$, and unit arrow in (B, B) given by $1_B = B \to B \rightleftharpoons B$, the collection of all m-regular involutive quantales as objects, and m-regular right Hilbert $A - B$ bimodules as arrows, forms a bicategory [Q^*].

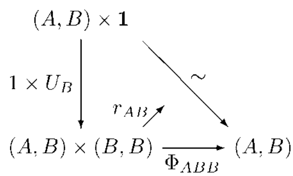
Proof: We have from Proposition 1.19 in Paseka (2001)



Similarly, from the Lemma 1.17 in Paseka (2001) we have



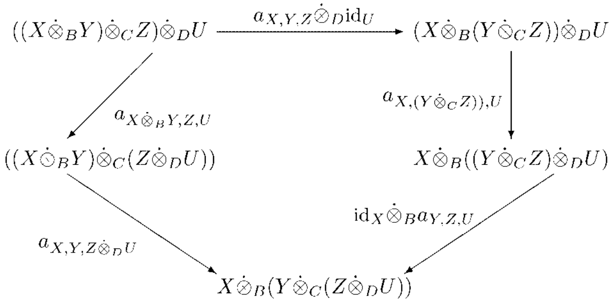
and from the Lemma 1.16 in Paseka (2001) we have



thus easily extending the above diagrams to 2-cells we have

$$\begin{array}{rccc} a_{X,Y,Z}: & (X\dot{\otimes}_B Y)\dot{\otimes}_C Z & \xrightarrow{\sim} & X\dot{\otimes}_B(Y\dot{\otimes}_C Z) \\ r_X: & X\dot{\otimes}_B 1_B & \xrightarrow{\sim} & X \\ l_X: & 1_A\dot{\otimes}_A X & \xrightarrow{\sim} & X. \end{array}$$

Again, from Paseka (2001) we have that the following diagrams commute:



and

$$\begin{array}{ccc} (X\dot{\otimes}_B B)\dot{\otimes}_B Y & \xrightarrow{a_{X,B,Y}} & X\dot{\otimes}_B(B\dot{\otimes}_B Y) \\ & {\scriptstyle r_X\dot{\otimes}_B \mathrm{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_X\dot{\otimes}_B l_Y} & \\ & X\dot{\otimes}_B Y. & \end{array}$$

□

Remark 3.4. An (m-regular) right Hilbert $A - B$ bimodule may be seen as a generalization of a (nondegenerate) *-homomorphism $\rho : A \to B$, for given such a ρ one constructs an (m-regular) right Hilbert $A - B$ bimodule B ($A \to B \rightleftharpoons B$) by $a \bullet b = \rho(a) \cdot b$, and the other operations as in Example 2.2. We write also $A \to B \rightleftharpoons B$.

Thus one obtains a contravariant faithful functor $\mathcal{F}$ from the category of $\mathcal{Q}u$ of m-regular involutive quantales with nondegenerate *-homomorphisms as arrows into the bicategory [$\mathbf{Q}^*$]. Namely, $\mathcal{F}(A) = A$ and $\mathcal{F}(\rho) = A \xrightarrow{\rho} B \rightleftharpoons B$. Clearly, $\mathcal{F}(\mathrm{id}_A) = 1_A = A \to A \rightleftharpoons A$ and again from the Lemma 1.16 in Paseka (2001) we have $\mathcal{F}(\rho_2 \circ \rho_1) = A \xrightarrow{\rho_2\circ\rho_1} C \rightleftharpoons C = (A \xrightarrow{\rho_1} B \rightleftharpoons B) \dot{\otimes}_B (B \xrightarrow{\rho_2} C \rightleftharpoons C)$.

Morita's theorems give a necessary and sufficient condition for the representation categories of two m-regular involutive quantales to be equivalent. In the present language, Morita theory starts as follows.

Theorem 3.5. (Paseka, 2002). *Two m-regular involutive quantales are isomorphic objects in the bicategory [$\mathbf{Q}^*$] iff they have equivalent representation categories (where the equivalence functor is required to be sup-preserving *-functor).*

Proof: Let A, B be m-regular involutive quantales. Then A and B are isomorphic in [$\mathbf{Q}^*$] iff there are an m-regular right Hilbert $A – B$ bimodule $A \to M \rightleftharpoons B$ and an m-regular right Hilbert $B – A$ bimodule $B \to N \rightleftharpoons A$ and such that $M\dot{\otimes}_B N \cong A$ and $N\dot{\otimes}_A M \cong B$. Then we may construct a functor $\mathcal{G} : \mathrm{Rep}\,(B) \to \mathrm{Rep}\,(A)$ by taking interior tensor products (see Paseka, 2001): on objects one has $\mathcal{G}_0(Y) = Y\dot{\otimes}_B N \in \mathrm{Rep}\,(A)_0$ for $Y \in \mathrm{Rep}\,(B)_0$, and on arrows one puts, in obvious notation, $\mathcal{G}_1(f) = f \dot{\otimes}_B \mathrm{id}_N$.

To go in the opposite direction, one repeats the above procedure, in defining a functor $\mathcal{F} : \mathrm{Rep}\,(A) \to \mathrm{Rep}\,(B)$ by means of $\mathcal{F}_0(X) = X\dot{\otimes}_A M$, etc. The proof is finished by applying the Theorem 2.4. in [Paseka (2002)]. □

Remark 3.6. An equivalence functor $\mathcal{F} : \mathrm{Rep}\,(A) \to \mathrm{Rep}\,(B)$ is automatically fibered, in the following sense. For each fixed m-regular involutive quantale C, the functor $\mathcal{F}$ defines an equivalence $\mathcal{F}_C$ between the categories (A, C) and (B, C), natural in C.

Naturality here means that, for any m-regular involutive quantales C, C' and nondegenerate involutive quantale homomorphisms $\varphi : C \to C'$, one has

$\varphi_B \circ \mathcal{F}_C = \mathcal{F}_{C'\varphi A}$, where $\varphi_A : (A, C) \to (A, C')$ is the induced functor given by $X \mapsto X \dot{\otimes}_c C'$ and $f : X_1 \to X_2 \mapsto f \dot{\otimes}_C \mathrm{id}'_C : X_1 \dot{\otimes}_C C' \to X_2 \dot{\otimes}_C C'$.

We have the following proposition.

Theorem 3.7. *Two m-regular involutive quantales A, B are isomorphic in [$\mathbf{Q}^*$] iff for any m-regular involutive quantale C one has a category equivalence $(A, C) \simeq (B, C)$, natural in C.*

Proof: Let A, B be m-regular involutive quantales. Then A and B are isomorphic in [$\mathbf{Q}^*$] iff there are an m-regular right Hilbert $A - B$ bimodule $A \to M \rightleftharpoons B$ and an m-regular right Hilbert $B - A$ bimodule $B \to N \rightleftharpoons A$ and such that $M \dot{\otimes}_B N \cong A$ and $N \dot{\otimes}_A M \cong B$. For any m-regular involutive quantale C, there is a functor $\mathcal{F}_C : (A, C) \to (B, C)$ given by $A \to X_1 \rightleftharpoons C \mapsto N \dot{\otimes}_A X_1$, $\alpha \mapsto \mathrm{id}_N \dot{\otimes}_A \alpha$ for all m-regular right Hilbert $A - C$ bimodules $A \to X_1 \rightleftharpoons C$, $A \to X_2 \rightleftharpoons C$ and all adjointable A-module maps $\alpha : X_1 \to X_2$ and there is a functor $\mathcal{G}_C : (B, C) \to (A, C)$ given by $B \to Y_1 \rightleftharpoons C \mapsto M \dot{\otimes}_B Y_1$, $\beta \mapsto \mathrm{id}_M \dot{\otimes}_B \beta$ for all m-regular right Hilbert $B - C$ bimodules $B \to Y_1 \rightleftharpoons C$, $B \to Y_2 \rightleftharpoons C$ and all adjointable B-module maps $\beta : Y_1 \to Y_2$ such that we have natural isomorphisms $\tau_C : \mathcal{G}_C \mathcal{F}_C \to \mathcal{I}d_{(A,C)}$ and $\sigma_C : \mathcal{F}_C \mathcal{G}_C \to \mathcal{I}d_{(B,C)}$. Conversely, if we have natural isomorphisms $\tau_C : \mathcal{G}_C \mathcal{F}_C \to \mathcal{I}d_{(A,C)}$ and $\sigma_C : \mathcal{F}_C \mathcal{G}_C \to \mathcal{I}d_{(B,C)}$, putting $C = A$ we have $N = \mathcal{F}_A(A)$ and putting $C = B$ we have $M = \mathcal{G}_B(B)$. The left B action on B is turned into a left A action on M by definition of $\mathcal{G}_B$, and the right B action on B is turned into a right B action on M through $\mathcal{G}_B$. Thus $M \in (A, B)$. Similarly, we have $\mathcal{F}_A(A) \in (B, A)$. The definition of equivalence of categories then trivially implies that $M^{-1} = \mathcal{F}_A(A)$. □

Proposition 3.8. *An m-regular right Hilbert bimodule $M \in (A, B)$ is invertible as an arrow in [$\mathbf{Q}^*$], so that A and B are isomorphic in [$\mathbf{Q}^*$], iff the nondegenerate involutive quantale homomorphism of A into $\mathcal{A}_B(M)$ of Definition* 2.1 *is an isomorphism $A \simeq \mathcal{K}_B(M)$. (If A has a unit, this isomorphism will be $A \simeq \mathcal{A}_B(M)$.)*

If $A \to M \rightleftharpoons B$ is invertible, its inverse (up to isomorphism) is $B \to M^* \rightleftharpoons A$, where M^* is the conjugate bimodule of M, on which B acts from the left by $b \bullet m = m \diamond b^*$ and a acts from the right by $n \diamond a = a^* \bullet n$. The A-valued inner product on M^* is given by $\langle m, n \rangle_A = \varphi^{-\mathbf{1}}(\Theta^B_{m,n})$, where $\varphi : A \to \mathcal{K}_B(M)$ is the pertinent isomorphism.

Proof: This is essentially Prop. 2.3 in [Schweizer (1999)] (Schweizer works with the category of C^*-algebras with equivalence classes of Hilbert bimodules as arrows, rather than with the bicategory whose arrows are the Hilbert bimodules themselves, but his proof may trivially be adapted to our situation). □

Remark 3.9. Note that, for any commutative C^*-algebra A, the corresponding lattice of closed ideals is a locally compact regular locale (frame) (Johnstone, 1982). Any locale is always an m-regular involutive quantale. So taking any locale that is not locally compact or regular we have an natural example of an involutive m-regular quantale that does not correspond to a C^*-algebra and to which results obtained before can applied.

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